On the gauge invariance of the non-Abelian Chern-Simons action for D-branes

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We present an elegant method to prove the invariance of the Chern-Simons part of the non-Abelian action for N coinciding D-branes under the R-R and NS-NS gauge transformations, by carefully defining what is meant by a background gauge transformation in the non-Abelian world volume action. We study as well the invariance under massive gauge transformations of the massive Type IIA supergravity and show that no massive dielectric couplings are necessary to achieve this invariance.[5]

It is well known by now that the physics of a set of N coincident Dp-branes can be very different from the physics of N parallel but separated Dp-branes. Witten showed [1] that in the former case a number of new massless states appear that can be arranged in representations of a $U(N)$ gauge group. In particular, the N Born-Infeld vectors form a single $U(N)$ Yang-Mills vector V_a and the transverse scalars, arranged in $N \times N$ matrices X^i, become non-Abelian matrices transforming in the adjoint representation of $U(N)$, where the I-th eigenvalue of the matrix X^i has the interpretation of the position of the I-th D-brane in the direction x^i.

The new physics associated to these extra massless string states has to be encoded in the world volume effective action, which now should be written in terms of the matrix valued fields V_a and X^i. Determining the exact form of the Born-Infeld action is a highly non-trivial problem, to which the solution is still not clear (see for instance [2]). A lot of progress has been made however over the last few years in the understanding of the structure of the non-Abelian Chern-Simons (or Wess-Zumino) action.

The first generalisation of the Chern-Simons term to the $U(N)$ case was proposed in [3]:

$$S_{\mathrm{D}p} \;=\; T_p \int P[C]\ \mathrm{Tr}\{e^{\mathcal{F}}\} \;=\; T_p \int \sum_n P[C_{p-2n+1}]\ \mathrm{Tr}\{\mathcal{F}^n\}. \tag{1}$$

Here the trace is taken over the Yang-Mills indices of the N-dimensional representation of $U(N)$ and $P[\Omega]$ denotes the pullback of the background field Ω to the world volume of the D-brane. The world volume field $\mathcal{F}$ is given by $\mathcal{F} = F + P[B]$, where $F = 2\partial V + i[V, V]$ is the non-Abelian field strength of the Born-Infeld vector and B the NS-NS two-form.

The invariance of this action under the gauge transformations of the background NS-NS and R-R fields was further investigated in [4], where it was shown as well that in order to be invariant under the massive gauge transformations of massive Type IIA supergravity [5, 6], extra m-dependent terms were needed in the action. These extra terms were also

[5] Talk given by B.J. at the XXVII Spanish Relativity Meeting in Madrid, September 2004.

obtained from the (massive) T-duality relations [6, 7] between the different D-brane actions, generalising to the non-Abelian case the Abelian calculation of [7].

Nowadays we know, however, that the Chern-Simons action for coincident D-branes presented in [4] is not the complete story. On the one hand, in the non-Abelian case the background fields in (1) must be functionals of the matrix-valued coordinates X^i [8]. Explicit calculations of string scattering amplitudes [9] suggest that this dependence is given by a non-Abelian Taylor expansion

$$C_{\mu\nu}(x^a, X^i) = \sum_n \frac{1}{n!}\partial_{k_1}...\partial_{k_n} C_{\mu\nu}(x^a, x^i)|_{x^i=0}\ X^{k_1}...\ X^{k_n}. \tag{2}$$

On the other hand, in order to have invariance under $U(N)$ gauge transformations the pullbacks of the background fields into the world volume have to be defined in terms of $U(N)$ covariant derivatives $D_a X^\mu = \partial_a X^\mu + i[V_a, X^\mu]$, rather than partial derivatives [10, 11]. For instance[6],

$$P[C_2] = C_{\mu\nu}\ D_{[a} X^\mu D_{b]} X^\nu. \tag{3}$$

This, together with the symmetrised trace prescription [12], that we will denote by curly brackets $\{..\}$, assures the invariance of the action under $U(N)$ gauge transformations

$$\delta V_a = D_a \chi, \qquad\qquad \delta X^i = i[\chi, X^i]. \tag{4}$$

Finally, the most important modification to the action (1) is the presence of new dielectric couplings to higher order background field potentials, arising as a consequence of T-duality in non-Abelian actions [13, 14]. It was found that the full T-duality invariant form of the Chern-Simons action is given by:

$$S_{\mathrm{D}p} \ = \ T_p \int \Big\{ P[e^{(\mathrm{i}_X \mathrm{i}_X)}(Ce^B)]\ e^F \Big\}, \tag{5}$$

where $(\mathrm{i}_X C)_{\mu_1...\mu_n}$ denotes the interior product $X^{\mu_0} C_{\mu_0...\mu_n}$.

One should note however that the presence of $U(N)$ covariant pullbacks has consequences on the invariance under gauge transformations of the background fields. Let us look for example at the variation $\delta C_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]}$ of the term given in (3). Naively filling in the variation in the pullback yields:

$$\delta\{P[C_2]\} = \{P[\partial \Lambda_1]\} = \{\partial_\mu \Lambda_\nu\ D_{[a} X^\mu D_{b]} X^\nu\}. \tag{6}$$

In the Abelian limit this gauge variation is a total derivative, such that the Λ_1 gauge invariance is assured in the D1-brane Chern-Simons action. In the non-Abelian case however the variation is not a total derivative such that not even D1-branes with topologically trivial world volumes are described by a gauge invariant action. The same goes for the non-Abelian couplings present in (5): a pullback of a variational parameter of the form $\{P[(\mathrm{i}_X \mathrm{i}_X)\partial \Lambda_{n+1}]\}$ is by no means a total derivative.

It is clear from these examples that the question of how to perform background gauge transformations in the action (5) is far from obvious. As the gauge transformations themselves are given by supergravity and the form of the action is derived in various compatible ways, we can not change these (too much). The only way therefore to construct a gauge invariant action is to change the way these transformations are implemented in the action.

Let us first concentrate on the simplest case of the monopole terms in (5), setting for now, all dielectric couplings to zero. In order to have an action invariant under the background

[6] From now on instead of working in the static gauge we will write everything in a "diffeomorphism invariant" way, with the understanding that $U(N)$ covariant derivatives reduce to ordinary ones for X^μ lying in the world volume of the D-branes.

gauge transformations, we need to fulfill three conditions. First, it must be possible to write the variation as a total derivative, secondly, the variation has to be a scalar under $U(N)$ gauge transformations and finally, it has to reduce to the known case in the Abelian limit. Therefore we define the variation of the pullback of a R-R field C_p under the background gauge transformation $\delta C_p = \partial \Lambda_{p-1}$ as [15]:

$$\delta P[C_p]\Omega \;\equiv\; DP[\Lambda_{p-1}]\Omega \;=\; D_{[a_1|}(\Lambda_{\mu_2...\mu_p} D_{|a_2} X^{\mu_2}...D_{a_p]}X^{\mu_p})\Omega, \tag{7}$$

where Ω is any combination of world volume or pullbacked background fields and where it is understood that all $U(N)$-valued objects appear symmetrised (though not in a trace). In particular for the simplest case with $\Omega = 1$ we find that

$$\begin{aligned}\delta P[C_p] =&\; D_{[a_1|}(\Lambda_{\mu_2...\mu_p} D_{|a_2} X^{\mu_2}...D_{a_p]}X^{\mu_p}) \\ =&\; P[\partial\Lambda_{p-1}] \;+\; \tfrac{i}{2}(p-1)\; \Lambda_{\mu_1...\mu_{p-1}}[F_{[a_1 a_2}, X^{\mu_2}]D_{a_3}X^{\mu_3}...D_{a_p]}X^{\mu_p}.\end{aligned} \tag{8}$$

With this definition we see that the variation is not just the pullback of the gauge parameter, but contains as well a non-Abelian correction term proportional to $[F, X]$, since the covariant derivative D_{a_1} not only acts on the background gauge parameter Λ_{p-1}, but also on the covariant derivatives in the pullback. For the Abelian case, the correction term disappears and we recover the well-known gauge transformation for Abelian D-brane actions. Furthermore once we consider terms in the action and trace over all $U(N)$ indices in the symmetrised trace prescription the variation is in fact a total derivative:

$$\delta\{P[C_p]\} = \{DP[\Lambda_{p-1}]\} = \partial\{P[\Lambda_{p-1}]\}. \tag{9}$$

In general for the background gauge transformations $\delta C_p = \sum_n \partial\Lambda_{p-2n-1}B^n - m\Sigma B^{(p-1)/2}$, we define the pullbacks in the action to vary as

$$\delta P[C_p]\Omega = \sum_n DP[\Lambda_{p-2n-1}]P[B^n]\Omega \;-\; mP[\Sigma B^{(p-1)/2}]\Omega. \tag{10}$$

Similarly the non-Abelian version of the NS-NS gauge transformation $\delta B = \partial\Sigma$ is given by

$$\delta P[B]\Omega = 2DP[\Sigma]\Omega, \qquad\qquad \delta V = -P[\Sigma]. \tag{11}$$

Note that the Born-Infeld field transforms as well, such that the non-Abelian field strength $\mathcal{F} = F + P[B]$ is an invariant quantity, as should be expected from the Abelian case.

With these definitions, the computation of the gauge transformations of the action

$$\mathcal{L} = \Big\{\sum_n P[C_{p-2n+1}]\mathcal{F}^n + m\;\omega_{2n+1}\Big\} \tag{12}$$

is straightforward, since it formally reduces to the Abelian case. Note that an extra Chern-Simons-like term [4]

$$\omega_{2n+1} = \sum_k \; V(\partial V)^{n-k}[V,V]^k \tag{13}$$

had to be added to the action of the even D-branes, in order to assure the invariance under the massive gauge transformations. These terms are constructed in such a way that they transform under the Yang-Mills gauge transformations as a total derivative, and under the Σ transformations as

$$\delta\omega_{2n+1} = -\Sigma F^n, \tag{14}$$

and thus cancel the massive gauge transformation of the R-R background fields.

So far we have rederived the results of [4] on the gauge invariance of non-Abelian Chern-Simons actions, taking into account explicitly the $U(N)$ covariant pullbacks and the fact that the background fields are functionals of the non-Abelian coordinates X^μ. As we have seen this forces a precise definition for what we mean by gauge variation of a non-Abelian pullback. A consistency check of our definitions (10)-(11) is that the variation of the pullback of a R-R p-form should be T-dual to the variation of the pullback of a R-R $(p-1)$-form field. We will now check this and see that in this manner we can find a natural way to also prove the gauge invariance of the dielectric terms.

To show this let us define a R-R field $\tilde{C}_p$, being related to C_p via a gauge transformation $\tilde{C}_P = C_p + \partial \Lambda_{p-1}$. We then have on the one hand by definition (7) that

$$P[\tilde{C}_p] = P[C_p] + DP[\Lambda_{p-1}] \tag{15}$$

while on the other hand we know from [13] that by applying T-duality on $\tilde{C}_p$ we get (for simplicity we truncate for now to the "diagonal approximation" $g_{\hat{\mu}x} = B_{\hat{\mu}\hat{\nu}} = 0$)

$$\begin{aligned} P[\tilde{C}_p] \ &\rightarrow P[\tilde{C}_{p-1}] \ + \ iP[(\mathrm{i}_X \mathrm{i}_X)\tilde{C}_{p+1}] \\ &= P[C_{p-1}] \ + \ iP[(\mathrm{i}_X \mathrm{i}_X)C_{p+1}] \ + \ DP[\Lambda_{p-2}] + \ iDP[(\mathrm{i}_X \mathrm{i}_X)\Lambda_p] \end{aligned} \tag{16}$$

where we used that $\tilde{C}_{p-1}$ and $\tilde{C}_{p+1}$ are related to, respectively, C_{p-1} and C_{p+1} by the same type of background gauge transformation that relates $\tilde{C}_p$ to C_p. We then find that the pullback of the gauge parameter transforms under T-duality as

$$DP[\Lambda_p] \rightarrow \ DP[\Lambda_{p-1}] \ + \ \ i \ DP[(\mathrm{i}_X \mathrm{i}_X)\Lambda_{p+1}]. \tag{17}$$

In other words, the variation of the pullback of a R-R p-form potential goes under T-duality to the variation of the pullback of a R-R $(p-1)$-form potential plus the variation of the pullback of the first dielectric coupling term:

$$\delta P[C_p] \ \rightarrow \ \delta P[C_{p-1}] \ + \ i \ \delta P[(\mathrm{i}_X \mathrm{i}_X)C_{p+1}], \tag{18}$$

if we define:

$$\delta P[(i_X i_X)C_{p+1}] \equiv \partial P[(\mathrm{i}_X \mathrm{i}_X)\Lambda_p]. \tag{19}$$

The derivation with the full T-duality rules (beyond the diagonal approximation) is straightforward and not very enlightening, so we rather concentrate on the generalisation of the variation (10) for dielectric couplings, which can be derived in a similar way. Under general R-R gauge transformations, the dielectric terms vary as

$$\begin{aligned} \delta P[(\mathrm{i}_X \mathrm{i}_X)C_p] = \sum_n \Big(&DP[(\mathrm{i}_X \mathrm{i}_X)\Lambda_{p-2n-1}]P[B^n] + \ DP[\mathrm{i}_X \Lambda_{p-2n-1}]P[(\mathrm{i}_X B)B^{n-1}] \\ &+ DP[\Lambda_{p-2n-1}]P[(\mathrm{i}_X B)^2 B^{n-2}] + \ DP[\Lambda_{p-2n-1}]P[(\mathrm{i}_X \mathrm{i}_X B)B^{n-1}]\Big). \end{aligned} \tag{20}$$

Note that the inclusion factor $(\mathrm{i}_X \mathrm{i}_X)$ acts on the various background fields. Similarly, under massive gauge transformations, the dielectric terms transform as

$$\delta P[(\mathrm{i}_X \mathrm{i}_X)C_p] = - \ m \ P[(\mathrm{i}_X \mathrm{i}_X)(\Sigma B^{(p-1)/2})]. \tag{21}$$

As an example let us now look at the gauge transformations of the non-Abelian action for D6-branes, being the simplest non-trivial case in which both dielectric couplings and massive gauge transformations are present. For this case, the non-Abelian Chern-Simons action can be written as

$$\mathcal{L}_{D6} \sim \Big\{ \sum_n P\Big[(\mathrm{i}_X \mathrm{i}_X) \mathcal{A}_{9-2n} \Big] F^n \Big\}, \tag{22}$$

where the p-forms $\mathcal{A}_p$ are defined as $\mathcal{A}_p = \sum_k C_{p-2k} B^k$. It is obvious from the Abelian case that each $\mathcal{A}_p$ is invariant under the R-R and massive gauge transformations, such that the invariance of the action (22) under the transformations (10), (20) and (21) is straightforward. It is also clear that besides the massive terms (13), introduced in [4], no other dielectric mass terms are needed to assure gauge invariance. This can also be confirmed by deriving the action by performing massive T-dualities from the D9-brane action [15].

The invariance under the NS-NS transformations (11) is however more subtle, due to the fact that $(\mathrm{i}_X \mathrm{i}_X)$ acts on B but not on F, so that they do not combine in an obvious way into the interior product of the gauge invariant field strength $\mathcal{F}$. In order to show the invariance under these transformations let us rewrite (22) as a function of the C_p, rather than $\mathcal{A}_p$, similar to the form of the action used in (1):

$$\begin{aligned} \mathcal{L}_{D6} \sim \Big\{ \sum_n P\Big[& (\mathrm{i}_X \mathrm{i}_X) C_{9-2n} \mathcal{F}^n \ + \ (\mathrm{i}_X C_{9-2n})(\mathrm{i}_X B) \mathcal{F}^{n-1} \\ & + C_{9-2n} (\mathrm{i}_X B)(\mathrm{i}_X B) \mathcal{F}^{n-2} \ + \ C_{9-2n} (\mathrm{i}_X \mathrm{i}_X B) \mathcal{F}^{n-1} \Big] \Big\}, \end{aligned} \tag{23}$$

Again here the inclusion terms $(\mathrm{i}_X \mathrm{i}_X)$ act both on C as on B. Note that all the B fields that are not acted upon by an inclusion term combine with the BI field strength F into the gauge invariant $\mathcal{F}$. However, the B's contracted with one or more i_X do not combine in a gauge invariant quantity and their variation can not be canceled by any other term in the action. The only field that also transforms under NS-NS transformations is the BI vector V_a, but for being a worldvolume fields it will never appear contracted with i_X.

In [15] it was suggested that the variation of these terms is identically zero, due to the fact that translations in the transverse directions are isometries. Recall that the action for non-Abelian Dp-branes is derived from the action for coincident D9-branes using T-duality [13], so that the directions in which the T-dualities are performed have to be isometric and hence the contractions of $\partial\Sigma$ with the transverse scalars must vanish, guaranteeing thus the gauge invariance of (23). Furthermore it was suggested in [15] that since in the non-Abelian case there is no clear notion of general coordinate transformations (see for example [16]-[19]), it is not clear how the resulting isometries can be removed.

However, there are now reasons to believe that the reasoning on [15] might not be completely correct, as phenomena such as the dielectric effect depend explicitly on the coordinate dependence in the transverse directions. It has been suggested[7] that the variation of the $\mathrm{i}_X B$ terms might be canceled by variations of other fields in the action that have not been taken into account yet. Indeed, it is not difficult to see that, after applying T-duality in a worldvolume direction x, the gauge variation of the x-component of V leads to the following transformation of the new transverse scalar in the T-dualised action:

$$\delta X^x = \xi^x + i\Sigma_\mu [X^x, X^\mu], \tag{24}$$

where ξ^x is the T-dual of Σ_μ and plays (in the Abelian case) the role of a coordinate transformation, while the second terms suggest a kind of non-Abelian NS-NS gauge variation for the embedding scalars X.

At this stage it is not clear what the interpretation of the variation (24) is (whether a coordinate transformation, or a gauge transformation) and whether it can be used to cancel the variations of the $(\mathrm{i}_X B)$ and $(\mathrm{i}_X \mathrm{i}_X B)$ terms in (23), but it does suggest that it might be helpful to use the well-known relation between NS-NS gauge transformations and coordinate transformations through T-duality in order to learn more about the problem

[7] We thank Rob Myers for this comment.

of general covariance of non-Abelian actions. We hope to report further progress in this direction soon [20].

Acknowledgments

We wish to thank Mees de Roo, Martijn Eenink and Rob Myers for the useful discussions. The work of J.A. is done as Aspirant F.W.O. She is also partially supported by the F.W.O.-Vlaanderen project G0193.00N and by the Belgian Federal Office for Scientific, Technical and Cultural Affairs through the Interuniversity Attraction Pole P5/27. The work of J.G. has been supported in part by the F.W.O.-Vlaanderen as postdoctoral researcher and by the Italian M.I.U.R. under the contract P.R.I.N. 2003023852. The work of B.J. is done as part of the program Ramón y Cajal of the M.E.C. (Spain). He was also partially supported by the M.E.C. under contract FIS 2004-06823 and by the Junta de Andalucía group FQM 101. The work of Y.L. has been partially supported by CICYT grants BFM2000-0357 and BFM2003-00313 (Spain). J.A., J.G. and Y.L. are also partially supported by the European Commission FP6 program MRTN-CT-2004-005104 in which Y.L. is associated to Universidad Autónoma de Madrid.

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